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ON THE APPROXIMATION OF POSITIONAL CONTROL PROBLEMS*

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Conditions for the approximation of positional control problems for contrallable systems of sufficiently general form are discussed, covering, in particular, certain classes of distributed-parameter plants. The constructions are based on the results in /1—6/. Analogous questions for programmed control problems were examined, for instance, in /7—14/, and for positional control problems, in /1,6/. In contrast to the approximation scheme proposed by Krasovskii for general evolution systems /1/, the resolving strategies in the present paper do not depend upon the accuracy ε. In contrast to /6/ a more general case is examined inclusing, in particular, approximation by the Galerkin method and by the difference method of straight lines.

1. We assume that a contrallable dynamic system Σ is defined, on the time interval $T = [t_0, \vartheta]$, $t_0 < \vartheta$, and in a metric space X (with metric ρ), by an operator Y associating with each $(t_1, t_2] \subset T$, $x \in X$, $u \in P(t_1, t_2]$, $v \in Q(t_1, t_2]$ a unique element $Y(t_1, x, t_2, u, v) \in X$. Here $P(t_1, t_2]$ ($Q(t_1, t_2]$) is the set of first (second) player's actions, admissible on $(t_1, t_2]$ /1/, with the properties: the actions $u(v)$ on $(t_1, t_2]$ define for every $\tau \in (t_1, t_2)$ the actions $u(t_1, \tau]$ ($v(t_1, \tau]$) and $u(\tau, t_2]$ ($v(\tau, t_2]$) on $(t_1, \tau]$ and $(\tau, t_2]$, respectively, and, conversely, the actions $u_1(v_1)$ on $(t_1, \tau]$ and the actions $u_2(v_2)$ on $(\tau, t_2]$ define the actions $u_1 + u_2$ ($v_1 + v_2$) on $(t_1, t_2]$. Operator Y possesses the semigroup property

$$Y(t_1, x, t_2, u_1 + u_2, v_1 + v_2) = Y(\tau, Y(t_1, x, \tau, u_1, v_1), t_2, u_2, v_2)$$

A rule $U(V)$ associating with each triple $\{t_1, t_2, x\}$, $t_0 \leqslant t_1 < t_2 \leqslant \vartheta$, $x \in X$, an element $U(t_1, t_2, x) \in P(t_1, t_2]$ ($V(t_1, t_2, x) \in Q(t_1, t_2]$) is called the first (second) player's strategy. A function $\varphi: [t_*, \vartheta] \to X$ with the properties

$$\varphi(t_*) = x_*, \quad \text{for} \quad t \in (\tau_j, \tau_{j+1}], \ j = 0, \ldots, m-1, \quad \varphi(t) = Y(\tau_j, \varphi(\tau_j), t, u_j(\tau_j, t], v_j(\tau_j, t])$$

where

$$u_j = U(\tau_j, \tau_{j+1}, \varphi(\tau_j)), \quad v_j \in Q(\tau_j, \tau_{j+1}], \qquad (u_j \in P(\tau_j, \tau_{j+1}], \ v_j = V(\tau_j, \tau_{j+1}, \varphi(\tau_j)))$$

is called a motion from position $\{t_*, x_*\}$, $t_* \in T$, $x_* \in X$, corresponding to strategy $U(V)$ and to the partitioning $\Delta = \{\tau_j, \ j = 0, \ldots, m(\Delta) \mid t_* = \tau_0 < \ldots < \tau_m = \vartheta\}$ of the interval $[t_*, \vartheta]$. The set of all motions corresponding to an initial position $\{t_*, x_*\}$, a strategy $U(V)$, and a partitioning Δ is denoted by the symbol $D(t_*, x_*, U(V), \Delta)$. In the space $T \times X$ of positions we introduce the metric $\sigma(\{t, x\}, \{\tau, y\}) = (|t - \tau|^2 + \rho(x, y)^2)^{1/2}$ and, if M is a set from $T \times X$, then

$$\sigma(\{t, x\}, M) = \inf_{\{\tau, y\} \in M} \sigma(\{t, x\}, \{\tau, y\})$$

Let sets M and N from $T \times X$ be specified.

Problem 1.1. For given $\{t_0, x_0\}, M, N$ find a strategy U with the property: for any $\varepsilon > 0$ a $\delta > 0$ can be found such that for every $\varphi \in D(t_0, x_0, U, \Delta)$, $d\Delta \leqslant \delta$ we can find $t \in T$ for which

$$\sigma(\{t, \varphi(t)\}, M) \leqslant \varepsilon \tag{1.1}$$

$$\sigma(\{\tau, \varphi(\tau)\}, N) \leqslant \varepsilon, \quad t_0 \leqslant \tau \leqslant t \tag{1.2}$$

Problem 1.2. For given $\{t_0, x_0\}, M, N$ find a strategy V with the property: there exist $\varepsilon > 0$, $\delta > 0$ such that the contact condition (1.1), (1.2) is excluded for every $\varphi \in D(t_0, x_0, V, \Delta)$, $d\Delta \leqslant \delta$. Here $d\Delta = \max\{\tau_{j+1} - \tau_j \mid j = 0, \ldots, m-1\}$.

2. Assume that a certain sequence $\{\Sigma_i\}$ of controllable dynamic systems exists in the sense of the above-mentioned definition. System Σ_i is specified on interval T in a metric space X_i (with metric ρ_i) with a semigroup operator Y_i mapping each $(t_1, t_2] \subset T$, $x_i \in X_i$, $u \in P(t_1, t_2]$, $v \in Q(t_1, t_2]$ into X_i. We construct an auxiliary sequence $\{\Sigma_i^*\}$ of controllable dynamic systems. System Σ_i^* is specified on T in the metric space $X_i^* = X_1 \times \ldots \times X_i$ (with

*Prikl.Matem.Mekhan.,44,No.6,1010-1018, 1980 a natural metric ρ_i^*) with an operator $Y_i^* = \{Y_1, \ldots, Y_i\}$ associating with each $(t_1, t_2] \subset T$, $x_i^* = \{x_1, \ldots, x_i\}$, $u \in P(t_1, t_2]$, $v \in Q(t_1, t_2]$ an element

$$Y_i^*(t_1, x_i^*, t_2, u, v) = \{Y_1(t_1, x_1, t_2, u, v), \ldots, Y_i(t_1, x_i, t_2, u, v)\}$$

Let a sequence

$$\{x_{0i}\},\ x_{0i} \in X_i,\ i = 1, 2, \ldots$$

of initial states be prescribed.

Condition 1. For every number i there exists an operator $A_i : X_i \to X$ with the properties:

1) $\rho(A_i x_{0i}, x_0) \to 0$ as $i \to \infty$ and

$$\rho(A_i Y_i(t_0, x_{0i}, t, u(t_0, t], v(t_0, t]), Y(t_0, x_0, t, u(t_0, t], v(t_0, t])) \to 0 \quad \text{as} \quad i \to \infty$$

uniformly in $t \in (t_0, \vartheta]$, $u \in P(t_0, \vartheta]$, $v \in Q(t_0, \vartheta]$;

2) operator A_i is uniformly continuous on the set

$$\bigcup_{t \in T} D_i(t),\quad D_i(t) = \{\varphi(t) \mid \varphi \in D_i = \bigcup_{(U_i)} \bigcup_{(\Delta)} D_i(t_0, x_{0i}, U_i, \Delta)\}$$

Condition 2. For every number i the set D_i of functions is equicontinuous in $t \in T$.

Condition 3. For every number i the system Σ_i^* is regular, i.e., a function $\mu_i : [0, \infty) \to [0, \infty)$ and a number $L_i \geqslant 0$ exist, with the properties:

1) $\mu_i(\gamma) \to 0$ as $\gamma \to 0$;

2) for any $t_1 < t_2$, x_i^* and y_i^* from $D_i^*(t_1)$ there exists $u^* \in P(t_1, t_2]$ ($v^* \in Q(t_1, t_2]$) such that for every $v \in Q(t_1, t_2]$ ($u \in P(t_1, t_2]$) we can find $v_* \in Q(t_1, t_2]$ ($u_* \in P(t_1, t_2]$) with the properties: for every $u \in P(t_1, t_2]$ ($v \in Q(t_1, t_2]$)

$$\rho_i^*(Y_i^*(t_1, x_i^*, t_2, u^*(u), v(v^*)), Y_i^*(t_1, y_i^*, t_2, u(u_*), v_*(v)))^2 \leqslant \rho_i^*(x_i^*, y_i^*)^2 \cdot e^{L_i \cdot (t_2 - t_1)} + \mu_i(t_2 - t_1) \cdot (t_2 - t_1)$$

An action u^* (v^*) is called an extremal action of the first (second) player, corresponding to the collection $\{t_1, t_2, x_i^*, y_i^*\}$. Without loss of generality we reckon that for every i the function μ_i is monotonic and that the inequalities

$$\mu_i(\gamma) \leqslant \mu_{i+1}(\gamma),\ L_i \leqslant L_{i+1}$$

are valid. For the sequence $\{x_{0i}\}$ of initial states there exists, by virtue of Condition 1, a positive sequence $\{\alpha_i\}$, $\alpha_i \to 0$ as $i \to \infty$, such that $\rho(A_i x_{0i}, x_0) \leqslant \alpha_i$ and for all $t \in (t_0, \vartheta]$, $u \in P(t_0, \vartheta]$, $v \in Q(t_0, \vartheta]$

$$\rho(A_i Y_i(t_0, x_{0i}, t, u(t_0, t], v(t_0, t]), Y(t_0, x_0, t, u(t_0, t], v(t_0, t])) \leqslant \alpha_i$$

Without loss of generality this sequence can be assumed monotonic.

We choose an arbitrary sequence $\{\eta_i\}$, $\eta_i > 0$ and $\eta_i \to 0$ as $i \to \infty$. For a given i, from the number $\eta_i / 3$ we determine by Condition 2 a number ξ_i for which the corresponding values of the functions from D_i, for values of argument t differing by the amount ξ_i, will differ in metric ρ_i by the amount $\eta_i / 3$. We choose an arbitrary sequence $\{\varepsilon_i\}$, $\varepsilon_i > 0$, $\varepsilon_i \geqslant \varepsilon_{i+1} \geqslant \ldots$, $\varepsilon_i \to 0$ as $i \to \infty$, $2\varepsilon_i \leqslant \min\{\xi_i, \eta_i/3\}$. Let Z_i be the set of all pairs $(u, v) \in P(t_0, \vartheta]\, Q(t_0, \vartheta]$, such that the function $\psi : T \to X$ defined by the relations $\psi(t_0) = A_i x_{0i}$, $\psi(t) = A_i Y_i(t_0, x_{0i}, t, u(t_0, t], v(t_0, t])$, $t_0 < t \leqslant \vartheta$ has the properties: there exists $t_* \in T$ such that

$$\sigma(\{t_*, \psi(t_*)\}, M) \leqslant \alpha_i + \varepsilon_i, \quad \sigma(\{\tau, \psi(\tau)\}, N) \leqslant \alpha_i + \varepsilon_i, \quad t_0 \leqslant \tau \leqslant t_*$$

We define the sets

$$M_i = \{\{t_*, x_i\} \mid x_i = Y_i(t_0, x_{0i}, t_*, u(t_0, t_*], v(t_0, t_*]) \text{ for } t_* > t_0 \text{ and } x_i = x_{0i} \text{ for } t_* = t_0\}$$

$$N_i = \{\{t, x_i\} \mid t_0 \leqslant t \leqslant t_*,\ x_i = Y_i(t_0, x_{0i}, t, u(t_0, t], v(t_0, t]) \text{ for } t > t_0 \text{ and } x_i = x_{0i} \text{ for } t = t_0\}$$

$$M_i^*(t) = M_1(t) \times \ldots \times M_i(t),\quad N_i^*(t) = N_1(t) \times \ldots \times N_i(t),\ (u, v) \in Z_i$$

Condition 4. For every number i and for $t \in T$ an operator $A_i^*(t) : X \to X_i^*$ exists with the properties:

1) a number $\beta_i > 0$ exists such that $\rho(x, y) \geqslant \beta_i \rho_i^*(A_i^*(t)x, A_i^*(t)y)$ for all $t \in T$, x and y from X;

2) $A_i^*(t_0) x_0 = x_{0i}^*$ and

$$A_i^*(t)\, Y(t_0, x_0, t, u(t_0, t], v(t_0, t]) = Y_i^*(t_0, x_{0i}^*, t, u(t_0, t], v(t_0, t])$$

for all $t \in (t_0, \vartheta]$, $u \in P(t_0, \vartheta]$, $v \in Q(t_0, \vartheta]$;

3) $A_i^*(t) M(t) \subseteq M_i^*(t)$, $A_i^*(t) N(t) \subseteq N_i^*(t)$, $t \in T$.

Analogs of Problem 1.1 and 1.2 with sets M_i^* and N_i^* can be examined for system Σ_i^*;

they will be referred to as Problems 1.1_i and 1.2_i.

Theorem 2.1. Let $\{t_0, x_0\}, M, N, \{t_0, x_{0i}\}, i = 1, 2, \ldots$ be specified. Then:

1) from Conditions 1—3 follows: Problem 1.1 is solvable if and only if Problem 1.1_i is solvable for every sufficiently large i ;

2) from Conditions 1—4 follows: either Problem 1.1 or Problem 1.2 is solvable for system Σ ;

3) from the Conditions 1—3 and statement 2) follows: Problem 1.2 is solvable if and only if Problem 1.2_i is solvable for at least one number i.

The theorem's proof relies on the construction of a sequence of singular Liapunov functionals λ_i: $t \times D(t) \times X_i^* \to [0, \infty)$, $i = 1, 2, \ldots$, /2—5/. Before we indicate the strategies resolving the problems we introduce certain additional definitions. By $F(t_1, x, t_2, y)$, where $t_0 \leqslant t_1 < t_2 \leqslant \vartheta$, $x, y \in X$, we denote the set of all pairs $(u, v) \in P(t_1, t_2] \times Q(t_1, t_2]$ such that $Y(t_1, x, t_2, u, v) = y$. We define the functional $\lambda_i : \lambda_i(t_0, x_0, x_i^*) = \rho_i^*(x_{0i}^*, x_i^*)^2$ and, for $t \in (t_0, \vartheta]$,

$$\lambda_i(t, x, x_i^*) = \inf_{F(t_0, x_0, t, x)} \rho_i^*(Y_i^*, (t_0, x_{0i}^*, t, u, v), x_i^*)^2$$

From Condition 3 follows: for any $t_1 < t_2$, $x \in D(t_1)$, $y_i^* \in D_i^*(t_1)$

$$\lambda_i(t_2, Y(t_1, x, t_2, u^*(u), v(v^*)), Y_i^*(t_1, y_i^*, t_2, u(u_*), v_*(v))) \leqslant \lambda_i(t_1, x, y_i^*)\cdot e^{L_i\cdot(t_2-t_1)} + \mu_i(t_2 - t_1)\cdot(t_2 - t_1)$$

where the actions u^*, v^*, u_*, v_* are determined from Condition 3 with $x_i^* = Y_i^*(t_0, x_{0i}^*, t_1, u, v)$, $(u, v) \in F(t_0, x_0, t_1, x)$.

A strategy U defined as follows is said to be extremal to the set sequence $\{W_i\}, W_i \subseteq T \times X_i^*$. Let

$$\varkappa(t, x) = \inf\left\{\frac{1}{i} \mid W_i(t) \neq \emptyset, \ \frac{1}{i} > (1 + \vartheta - t)\cdot\lambda_i(t, x, W_i(t))\cdot e^{L_i\cdot(\vartheta-t)}\right\}, \quad \lambda_i(t, x, W_i(t)) = \inf_{z \in W_i(t)} \lambda_i(t, x, z)$$

For a triple $\{t_1, t_2, x\}$, $t_1 < t_2$, $x \in D(t_1)$, such that the function $\varkappa(t_1, x)$ is defined, we assume:

1) $i_0 = i_0(t_1, t_2, x)$ is some number such that

$$\varkappa(t_1, x) \leqslant \frac{1}{i_0} < \varkappa(t_1, x) + (t_2 - t_1)^2, \qquad \frac{1}{i_0} > (1 + \vartheta - t_1)\cdot\lambda_{i_0}(t_1, x, W_{i_0}(t_1))\cdot e^{L_{i_0}\cdot(\vartheta-t_1)}$$

2) $z = z(t_1, t_2, x)$ is an element of $W_{i_0}(t_1)$ such that

$$\lambda_{i_0}(t_1, x, z) \leqslant \lambda_{i_0}(t_1, x, W_{i_0}(t_1)) + \mu_{i_0}(t_2 - t_1)\cdot(t_2 - t_1)$$

3) $U(t_1, t_2, x)$ is the first player's extremal action corresponding to the collection $\{t_1, t_2, Y_{i_0}^*(t_0, x_{0i_0}^*, t_1, u, v), z\}$, $(u, v) \in F(t_0, x_0, t_1, x)$.

For a triple $\{t_1, t_2, x\}$, $t_1 < t_2$, such that $x \notin D(t_1)$, because the function $\varkappa(t_1, x)$ is not defined, we assume: $U(t_1, t_2, x)$ is an arbitrary element of $P(t_1, t_2]$. The second player's extremal strategy is defined analogously.

From Conditions 2 and 3 for system Σ_i^* there follows an alternative in the game consisting of Problems 1.1_i and 1.2_i. Problem 1.1_i is solvable if and only if

$$x_{0i}^* \in \bigcap_{\varepsilon>0} Z_\varepsilon^i(t_0)$$

where Z_ε^i is the set of positional ε-absorptions of set M_i^* inside N_i^* /2—5/. The set Z_ε^i (if it is not empty) is a u-stable bridge of system Σ_i^* /2—5/, contained in $N_i^{*\varepsilon}$ (the closed ε-neighborhood of N_i^*) and staying in $M_i^{*\varepsilon}$ (the closed ε-neighborhood of M_i^*). The proof of these assertions is based on the ideas in /1—5/.

The assertion in the first part of Theorem 2.1 follows from

Lemma 2.1. Let $\{t_0, x_0\}, M, N, \{t_0, x_{0i}\}$ $(i = 1, 2, \ldots)$ be specified and Conditions 1—3 be fulfilled. Let $W_i = \{\{t, x_i^*\} \mid, t \in T, x_i^* \in Z_{\varepsilon_i}^i(t) \cap D_i^*(t)\}$ and $x_{0i}^* \in W_i(t_0)$ for all sufficiently large i. Then the startegy U^i extremal to sequence $\{W_i\}$ solves Problem 1.1. If Problem 1.2_i is solvable for even one number i, then Problem 1.1 cannot be solved.

The assertion in the second part of Theorem 2.1 follows from the assertion in the first part, Condition 4, and the fact that if strategy V_i^* solves Problem 1.2_i for some i, then strategy $V(t_1, t_2, x) = V_i^*(t_1, t_2, A_i^*(t_1)x)$ solves Problem 1.2. The assertion in the third part of Theorem 2.1 follows from the preceding two.

3. We indicate a special case when we can manage without such a technical device as the construction of the auxiliary systems Σ_i^*. It is the following: systems Σ_i have the structure of systems Σ_i^*, i.e., a sequence $\{\Sigma_{*i}\}$ of controllable dynamic systems exists such that each system Σ_i is constructed from $\Sigma_{*1}, \ldots, \Sigma_{*i}$ similarly to how Σ_i^* was constructed from $\Sigma_1, \ldots, \Sigma_i$. The thing is that in this case it is sufficiently simple to construct a suitable operator $B_i : X_{i+1} \to X_i$, $i = 1, 2, \ldots$ with the properties

$$\rho_i(B_i x_{i+1}, B_i y_{i+1}) \leqslant \rho_{i+1}(x_{i+1}, y_{i+1}), \quad B_i Y_{i+1}(t_0, x_{0i+1}, t, u, v) = Y_i(t_0, x_{0i}, t, u, v)$$

Example 3.1. Let system Σ be described by the equations of oscillations of a homogeneous string with distributed controlling loads

$$z_{tt} = z_{\xi\xi} + b(\xi) u(t) + c(\xi) v(t), \quad z(t, 0) = z(t, l) = 0, \quad t \in T, \quad 0 < \xi < l, \quad z(t_0, \xi) = z_0(\xi), \quad z_t(t_0, \xi) = z_1(\xi) \tag{3.1}$$

Let $b, c, z_1 \in L_2(0, l)$, $z_0 \in W_2^{01}(0, l)$ (see /15/, for example, for the definitions of the spaces). Then, following /4,16/, we can take

$$X = W_2^{01}(0, l) \times L_2(0, l)$$

An admissible action $u(v)$ of the first (second) player on $(t_1, t_2]$ is the Lebesgue measurable function

$$u : (t_1, t_2] \to [-\mu, \mu] \; (v : (t_1, t_2] \to [-\nu, \nu]), \qquad Y(t_1, x, t_2, u, v) = \{z(t_2, \cdot; t_1, x, u, v), z_t(t_2, \cdot; t_1, x, u, v)\}$$

Here

$$z = z(t, \xi; \ldots), \; t_1 \leqslant t \leqslant t_2, \; 0 < \xi < l, \qquad z_t = z_t(t, \xi; \ldots), \; t_1 \leqslant t \leqslant t_2, \; 0 < \xi < l$$

are a generalized solution of problem (3.1) /16/ (where, naturally, we must set $z(t_1, \cdot) = x^{(1)}$, $z_t(t_1, \cdot) = x^{(2)}$, $x = \{x^{(1)}, x^{(2)}\}$, $t_1 \leqslant t \leqslant t_2$) and its generalized derivative with respect to t. As was shown in /16/

$$z \in C([t_1, t_2]; W_2^{01}(0, l)), \; z_t \in C([t_1, t_2]; L_2(0, l))$$

Let $\{q_i, \omega_i\}$ be a solution of the spectal problem

$$\omega_{\xi\xi} = -q\omega, \; \omega(0) = \omega(l) = 0$$

Obviously, $q_i = \pi^2 i^2 / l^2$, $\omega_i(\xi) = \sqrt{2/l} \sin(\pi i \xi / l)$, $i = 1, 2, \ldots$. Then we can set

$$X_i = R^i \times R^i, Y_i(t_1, x_i, t_2, u, v) = \{z_i(t_2; t_1, x_i, u, v), z_i^{\cdot}(t_2; t_1, x_i, u, v)\}$$

where $z_i(\cdot; \ldots)$ is a solution almost everywhere of the system of ordinary differential equations

$$y_j^{\cdot\cdot} = -q_i y_j + \langle b, \omega_j \rangle u + \langle c, \omega_j \rangle v, \; j = 1, \ldots, i$$
$$y_j(t_1) = x_i^{1j}, \; y_j^{\cdot}(t_1) = x_i^{2j}, \; x_i = \{x_i^{(1)}, x_i^{(2)}\}$$
$$x_i^{(1)} = \{x_i^{11}, \ldots, x_i^{1i}\}, \; x_i^{(2)} = \{x_i^{21}, \ldots, x_i^{2i}\}$$

Here $\langle \cdot, \cdot \rangle$ is the scalar product in $L_2(0, l)$. The example given corresponds to the special case being examined. In order to satisfy the corresponding analogs of Conditions 1—4, we should take

$$x_{0i} = \{x_{0i}^{(1)}, \; x_{0i}^{(2)}\}, \; x_{0i}^{1j} = \langle x_0^{(1)}, \omega_j \rangle, \; x_{0i}^{2j} = \langle x_2^{(2)}, \omega_j \rangle, \quad j = 1, \ldots i; \; A_i x_i = \left\{\sum_{j=1}^{i} x_i^{1j} \omega_j, \; \sum_{j=1}^{i} x_i^{2j} \omega_j\right\}$$

$$\beta_i = 1, \; A_i^*(t) x = \{x_i^{(1)}, x_i^{(2)}\}, \qquad x_i^{1j} = \langle x^{(1)}, \omega_j \rangle, \quad x_i^{2j} = \langle x^{(2)}, \omega_j \rangle$$

The quantities $\mu_i, L_i, u^*, v^*, u_*, v_*$ are defined as in /2/ and as the sets M_i and N_i we can assume

$$M_i = \{\{t, x_i\} \mid t \in T, \; x_i = A_i^*(t) x, \; \{t, x\} \in M\}, \quad N_i = \{\{t, x_i\} \mid t \in T, \; x_i = A_i^*(t) x, \; \{t, x\} \in N\}$$

The problem can be analyzed analogously for a more general form of hyperbolic system /4/. A parabolic system was analyzed in /6/.

4. In conclusion, by example of an ε-encounter problem /1/ we show one of the methods of constructing the ε-strategy

$$U^\varepsilon(t_1, t_2, x), \; t_0 \leqslant t_1 < t_2 \leqslant \vartheta, \; x \in X, \; \varepsilon > 0$$

solving this problem.

An ε-encounter problem. Let $\{t_0, x_0\}$, M, N be specified. For any $\varepsilon > 0$ find an ε-strategy U^ε with the property: $\delta > 0$ can be found such that for any $\varphi \in D(t_0, x_0, U^\varepsilon, \Delta)$, $d\Delta \leqslant \delta$ we can find $t \in T$ for which (1.1) and (1.2) are valid.

Condition 5. For some sets M_i, N_i, $i = 1, 2, \ldots$, let

$$\sigma_i(\{t, \psi_i(t)\}, M_i(N_i)) \to \sigma(\{t, \psi(t)\}, M(N))$$

as $i \to \infty$ uniformly in $t \in T$, $u \in P(t_0, \vartheta]$, $v \in Q(t_0, \vartheta]$

$$\psi_i(t_0) = x_{0i}, \; \psi_i(t) = Y_i(t_0, x_{0i}, t, u(t_0, t], v(t_0, t])$$
$$\psi(t_0) = x_0, \; \psi(t) = Y(t_0, x_0, t, u(t_0, t], v(t_0, t]), \; t_0 < t \leqslant \vartheta$$

Condition 6. For all sufficiently large i a strategy U_i exists (possibly including among the arguments an auxiliary variable formable in a chain of controls, for example, a guide variable) which ensures the solution of Problem 1.1_i (for system Σ_i with sets M_i, N_i and initial position $\{t_0, x_{0i}\}$), stable with respect to noise in the measurement of state x_i.

Let $\xi_i = \xi_i(\varepsilon) > 0$ be the magnitude of admissible noise, corresponding to $\varepsilon > 0$ /2/.

Condition 7. Let Condition 6 be fulfilled, For any $\varepsilon > 0$, for all sufficiently large i an operator $C_i : X \to X_i$ exists such that

$$\rho_i (C_i \psi (t), \psi_i (t)) \leqslant \xi_i (\varepsilon)$$

at once for all $t \in T$, $u \in P(t_0, \vartheta]$, $v \in Q(t_0, \vartheta]$.

Theorem 4.1. Let $\{t_0, x_0\}, M, N, \{t_0, x_{0i}\}, M_i, N_i, i = 1, 2, \ldots$ be specified and let Conditions 5–7 be fulfilled. Then for any $\varepsilon > 0$ we can find a number $i_* = i_*(\varepsilon)$ such that for every $i \geqslant i_*$ the ε-strategy

$$U^\varepsilon (t_1, t_2, x) = U_i (t_1, t_2, C_i x)$$

solves the ε-encounter problem.

Example 4.1. Let system Σ be described by the heat conduction equation for a thin homogeneous rod with distributed controlling forces

$$z_t = z_{\xi\xi} + b(\xi) u(t) + c(\xi) v(t), \quad z(t, 0) = z(t, l) = 0, \ t \in T, \ 0 < \xi < l, \quad z(t_0, \xi) = z_0(\xi) \tag{4.1}$$

Let $b, c, z_0 \in W_2^{01}(0, l)$. Then following /4.15/, we can assume $X = W_2^{01}(0, l)$ with a metric of space $L_2(0, l)$; an admissible action $u(v)$ of the first (second) player on $(t_1, t_2]$ is the Lebesgue measurable function

$$u : (t_1, t_2] \to [-\mu, \mu] \ (v : (t_1, t_2] \to [-\nu, \nu]), \quad Y(t_1, x, t_2, u, v) = z(t_2, \cdot; t_1, x, u, v)$$

where $z = z(t, \xi; \ldots)$, $t_1 \leqslant t \leqslant t_2$, $0 < \xi < l$ is the generalized solution of problem (4.1) /15,16/(where, naturally, we should set $z(t_1, \cdot) = x$, $t_1 \leqslant t \leqslant t_2$). As was shown in /15,16/, $z \in C([t_1, t_2]; W_2^{01}(0, l))$.

On $[0, l]$ we prescribe the difference grid

$$\xi_k = kh, \ k = 0, \ldots, i+1, \ h(i+1) = l$$

then we can set: $X_i = R^i$ with metric

$$\rho_i(x_i, y_i) = \Big(h \sum_{j=1}^{i} (x_i^j - y_i^j)^2\Big)^{1/2}$$

$Y_i(t_1, x_i, t_2, u, v) = z_i(t_2; t_1, x_i, u, v)$, where $z_i(\cdot; \ldots)$ is a solution almost everywhere of the system of ordinary differential equations of the method of lines

$$z_i^{j\cdot} = ((z_i^j)_\xi)_{\bar\xi} + b(\xi_j) u + c(\xi_j) v, \ j = 1, \ldots, i, \quad z_i^j(t_1) = x_i^j, \ x_i = \{x_i^1, \ldots, x_i^i\} \tag{4.2}$$

Let M and N be bounded sets from $T \times W_2^{01}(0, l)$. In order to satisfy Condition 5 we set

$$M_i = \{\{t, x_i\} \mid t \in T, \ x_i = C_i x, \ \{t, x\} \in M\}, \quad N_i = \{\{t, x_i\} \mid t \in T, \ x_i = C_i x, \ \{t, x\} \in N\}, \quad C_i x = \{x(\xi_1), \ldots, x(\xi_i)\} \tag{4.3}$$

Obviously, Condition 6 satisfies the procedure of control with a guide if $x_{0i} \in W_i(t_0)$, where W_i is the set of positional absorptions of system (4.2) /2/. Finally, we note that for a given $\varepsilon > 0$ the quantity $\xi_i(\varepsilon) > 0$ can be taken as only one for all i; therefore, taking into account the convergence of the solution of problem (4.2) to the solution of problem (4.1), we can take operator (4.3) as the operator C_i.

For simplicity let phase constraints be absent and let $M = M(\vartheta)$. Then, allowing for the estimate of convergence rate in Condition 5 (of the order of $\gamma \cdot h^{1/2}$) and the estimate of the mismatch between the motion and the guide in Condition 6, in Theorem 4.1 as applied to the given case we can take

$$i \geqslant \gamma_1 / \varepsilon^2, \quad 0 < \delta < \gamma_2 \varepsilon / i^2$$

where $\gamma, \gamma_1, \gamma_2$ are positive constants explicitly determinable from z_0, b, c, μ, ν and M.

In the example given it is not difficult to verify Conditions 1–3 wherein as the operator A_i we can take the polynomial interpolation operator, while Condition 3 is satisfied as in /2/. For system (4.1) in the game consisting of Problems 1.1 and 1.2 an alternative is valid. The constructions carried out in this example can be extended to certain classes of hyperbolic systems.

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